Lecture 16

Ex: Suppose $z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$.
If $x(2,3) = 4$, $y(2,3) = 9$, $f_x(4,9) = -1$, $f_y(4,9) = \pi$
$x_u(2,3) = 1$, $x_v(2,3) = 6$, $y_u(2,3) = \sqrt{2}$, and $y_v(2,3) = -\pi$,

find $z_u(2,3)$ and $z_v(2,3)$.

Sol:

$Z_u = Z_x X_u + Z_y Y_u$, so

$$z_u(2,3) = Z_x(x(2,3), y(2,3))X_u(2,3) + Z_y(x(2,3), y(2,3))\, y_u(2,3)$$
$$= Z_x(4,9)\, x_u(2,3) + Z_y(4,9)\, y_u(2,3) = (-1)(1) + (\pi)(\sqrt{2}) = \sqrt{2}\pi - 1$$

With gradient: $\vec{G}(u,v) = \langle x(u,v), y(u,v) \rangle$

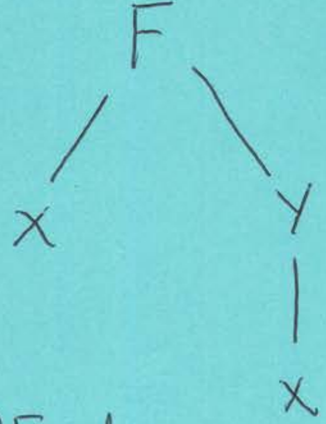
$$z_v(2,3) = \nabla f(\vec{G}(2,3)) \cdot \frac{\partial \vec{G}}{\partial v}(2,3) = \nabla f(4,9) \cdot \frac{\partial \vec{G}}{\partial v}(2,3)$$

$$= \langle -1, \pi \rangle \cdot \langle 6, -\pi \rangle = -6 - \pi^2$$

$\diamond$

Implicit Differentiation : Calc I :

Suppose a function $F(x,y)=0$ defines y implicitly as a function of x. Then we can find $\frac{dy}{dx}$ via the chain rule: take $\frac{\partial}{\partial x}$ of both sides:



$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0, \quad \text{but } \frac{dx}{dx} = 1, \text{ so}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Calc III :

We can do the same if z is defined implicitly as a function of x and y by $F(x,y,z)=0$.

In this case: $\frac{\partial z}{\partial x}$ is obtained by

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial x}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial y}{\partial x} = 0 \text{ since } y \text{ is independent of } x.$$

So, $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$. Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$

<u>Ex</u>: Find all first partials of z, where z is defined implicitly by: $e^{xz} + xy = 10$.

<u>Sol</u>: ~~Take $\frac{\partial}{\partial x}$ of both sides~~

Let $F(x,y,z) = e^{xz} + xy - 10$

Then $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\left(\frac{ze^{xz}+y}{xe^{xz}}\right)$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x}{xe^{xz}} = -e^{-xz}$$

$\diamond$

There are some technical assumptions involved in doing this implicit differentiation (indeed, for when z in fact is implicitly defined by $F=0$). This is encapsulated in the <u>implicit function theorem</u>.

We end this section with an application.

Ex: Suppose we have a box containing 42 cubic inches of an incompressible fluid. Holding the height of the box fixed at 7 inches, we squeeze the width of the box so that it decreases at a rate of 1 in/min. How fast is the length of the box changing when the width is 3 inches?

Sol: Volume $V = l \cdot w \cdot h$. We want $\frac{dl}{dt}$.

Take $\frac{d}{dt}$ of the volume equation:

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{dl}{dt} + lh\frac{dw}{dt} + lw\frac{dh}{dt}$$

$\frac{dV}{dt} = 0\ \text{in}^3/\text{m}$: fluid is incompressible, $\frac{dh}{dt} = 0\ \text{in}/\text{m}$: height is fixed

$\frac{dw}{dt} = -1\ \text{in}/\text{m}$: assumption. $V = 42\ \text{in}^3 = lwh = l(\overset{w}{3\ \text{in}})(\overset{h}{7\ \text{in}}) \Rightarrow l = 2\ \text{in}$

$$\Rightarrow 0\ \text{in}^3/\text{m} = (21\ \text{in}^2)\frac{dl}{dt} + (14\ \text{in}^2)(-1\ \text{in}/\text{m}) + (6\ \text{in}^2)(0\ \text{in}/\text{m}) \Rightarrow \frac{dl}{dt} = \frac{14\ \text{in}^3/\text{m}}{21\ \text{in}^2} = \frac{2}{3}\ \text{in}/\text{m}$$

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14.6 - Directional Derivatives

When we introduced partial derivatives, we talked about about them being in the x- or y-direction (f_x and f_y resp.). There are a myriad of other directions to move in, so a natural question is, "can we take derivatives in other directions?"

The answer is "yes." First pick a direction: this is best done by picking a unit vector $\vec{u}=\langle a,b\rangle$ in the direction you want, then the <u>directional derivative</u> of $f=f(x,y)$ in the direction $\vec{u}$ is

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ha, y+hb)-f(x,y)}{h} = \nabla f(x,y)\cdot \vec{u}}$$

If we write $\vec{x}=\langle x,y\rangle$, then we can write

$$D_{\vec{u}} f(\vec{x}) = \lim_{h\to 0} \frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$$

Of course, this generalizes to functions of three or more variables, e.g., if $\vec{u}=\langle a,b,c\rangle$ is a unit vector in $\mathbb{R}^3$,

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z)\cdot \vec{u}.}$$

<u>Ex</u>: Find the directional derivative of $f(x,y) = e^x \tan y$ at the point $(3, \frac{\pi}{3})$ in the direction of $\vec{v} = \langle 1, -2 \rangle$.

$\tan\frac{\pi}{3} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$

$\sec^2\frac{\pi}{3} = \left(\frac{1}{\left(\frac{1}{2}\right)}\right)^2 = 4$

<u>Sol</u>: First, $\nabla f(x,y) = \langle e^x \tan y, e^x \sec^2 y \rangle$

so $\nabla f(3, \frac{\pi}{3}) = \langle e^3 \cdot \sqrt{3}, e^3 (2)^2 \rangle = \langle e^3\sqrt{3}, 4e^3 \rangle$

⚠ We can't just take $\nabla f(3, \frac{\pi}{3}) \cdot \vec{v}$ for our directional derivative since $\vec{v}$ is not a <u>unit vector</u>!

Normalize $\vec{v}$: $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, -2 \rangle}{\sqrt{1^2 + (-2)^2}} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

Now, $D_{\hat{v}} f(3, \frac{\pi}{3}) = \langle e^3\sqrt{3}, 4e^3 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$$= e^3\sqrt{\frac{3}{5}} - \frac{8e^3}{\sqrt{5}}$$

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A natural question we can ask is "in what direction does f have the biggest rate of change?" Let $\vec{u}$ be a unit vector and θ the angle between ∇f and $\vec{u}$. Then,

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta$$

this is biggest when $\cos\theta = 1$, i.e., $\theta = 0$. Here $D_{\vec{u}} f = |\nabla f|$.